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BALANCED COHEN-MACAULAY COMPLEXES

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ABSTRACT. A balanced complex of type $(a_1, \ldots, a_m)$ is a finite pure simplicial complex Δ together with an ordered partition $(V_1, \ldots, V_m)$ of the vertices of Δ such that $\operatorname{card}(V_i \cap F) = a_i$ for every maximal face F of Δ. If $\mathbf{b} = (b_1, \ldots, b_m)$, then define $f_{\mathbf{b}}(\Delta)$ to be the number of $F \in \Delta$ satisfying $\operatorname{card}(V_i \cap F) = b_i$. The formal properties of the numbers $f_{\mathbf{b}}(\Delta)$ are investigated in analogy to the f-vector of an arbitrary simplicial complex. For a special class of balanced complexes known as balanced Cohen-Macaulay complexes, simple techniques from commutative algebra lead to very strong conditions on the numbers $f_{\mathbf{b}}(\Delta)$. For a certain complex $\Delta(P)$ coming from a poset P, our results are intimately related to properties of the Möbius function of P.

1. Introduction. We are concerned with the problem of obtaining information on the number $f_i = f_i(\Delta)$ of i-dimensional faces of a finite simplicial complex Δ. (All terminology is defined below.) There are two significant classes of complexes Δ for which a complete characterization of the numbers f_i, $0 \leqslant i \leqslant \dim \Delta$, has been obtained, viz., the class of all complexes and the class of Cohen-Macaulay complexes. Here we introduce a new class which we call balanced complexes. Balanced complexes possess invariants $f_{\mathbf{b}}$ more discriminating than the numbers f_i, and the formal properties of these invariants will be investigated. In the case of balanced Cohen-Macaulay complexes Δ, simple techniques from commutative algebra lead to conditions on the invariants $f_{\mathbf{b}}$ which are considerably stronger than those obtained merely by assuming Δ is Cohen-Macaulay. For a certain complex $\Delta(P)$ coming from a poset P, our results are intimately related to properties of the Möbius function of P.

We now proceed to the basic definitions and terminology. We employ the following notation throughout:

$\mathbf{N}$, set of nonnegative integers,

$\mathbf{P}$, set of positive integers,

$[n]$, $\{1, 2, \ldots, n\}$, where $n \in \mathbf{P}$,

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$T \subset S$, T is a subset of S, allowing $T = \emptyset$ or $T = S$,

$\mathbf{e}_i$, the ith unit coordinate vector in $\mathbf{N}^m$, i.e., $\mathbf{e}_i = (\varepsilon_1, \ldots, \varepsilon_m)$, where $\varepsilon_j = \delta_{ij}$.

Now let Δ be a simplicial complex, or *complex* for short, on the vertex set $V = \{x_1, \ldots, x_n\}$. Thus Δ is a collection of subsets of V satisfying the two conditions: (i) $\{x\} \in \Delta$ for all $x \in V$, and (ii) if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. If $F \in \Delta$ and F has $i + 1$ elements, then we call F an *i-face* of Δ and write $\dim F = i$. If $\delta = \max\{\dim F: F \in \Delta\}$, then we call Δ a *δ-complex* and write $\dim \Delta = \delta$. If every maximal face of Δ is a δ-face, then we say that Δ is *pure*. Let $f_i = f_i(\Delta)$ denote the number of i-faces of Δ. Thus $f_0 = n$. The vector $\mathbf{f} = \mathbf{f}(\Delta) = (f_0, f_1, \ldots, f_\delta)$ is called the *f-vector* of Δ. We will employ the notation $\Delta = \langle abc, bcd, bde \rangle$ to indicate that the maximal faces of Δ are $\{a, b, c\}$, $\{b, c, d\}$ and $\{b, d, e\}$. Hence the f-vector of this Δ is (5, 7, 3).

The problem often arises of obtaining information about the f-vectors of various complexes Δ. The first significant result along these lines, essentially due to Joseph Kruskal [**12**] and G. Katona [**11**] (see [**9**, §8] for an exposition), is an explicit characterization of those vectors $\mathbf{f} = (f_0, f_1, \ldots, f_\delta)$ which are the f-vectors of some complex Δ. We will call such vectors *K-vectors*. Kruskal and Katona actually only proved that the condition in Theorem 1.1 below is necessary; but the sufficiency of this condition is immediate from their proofs.

1.1 Theorem. *Given positive integers f and i, write*

$$f = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geqslant j \geqslant 1$ (such a representation exists and is unique), and define

$$f^{(i)} = \binom{n_i}{i+1} + \binom{n_{i-1}}{i} + \cdots + \binom{n_j}{j+1}.$$

Then the vector $\mathbf{f} = (f_0, f_1, \ldots, f_\delta)$ of positive integers is a K-vector if and only if $f_{i+1} \leqslant f_i^{(i+1)}$ for $0 \leqslant i \leqslant \delta - 1$. $\square$

One can now ask for special classes of complexes whose f-vectors have characterizations analogous to that of Theorem 1.1. One class for which such a characterization can be given consists of the so-called Cohen-Macaulay complexes [**23**]. These may be defined either algebraically or topologically; it is a basic result of G. Reisner [**16**] that the two definitions are equivalent. Both definitions will be of use to us here. First we define the notion of a Cohen-Macaulay ring in the case of interest to us. Let k be a field, and let m be a positive integer. Suppose R is a finitely-generated $\mathbf{N}^m$-graded k-algebra. I.e., the additive group of R can be written as a direct sum $R = \sum_{\alpha \in \mathbf{N}^m} R_\alpha$, where $R_{\boldsymbol{\alpha}} R_{\boldsymbol{\beta}} \subset R_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$, $R_{\mathbf{0}} = k$, and R is finitely generated as a k-algebra. If $x \in R_{\boldsymbol{\alpha}}$ we say that x is *homogeneous* of degree $\boldsymbol{\alpha}$, written $\deg x = \boldsymbol{\alpha}$. If we wish to emphasize that $\boldsymbol{\alpha} \in \mathbf{N}^m$, then we say that x is $\mathbf{N}^m$-homogeneous. It follows from the fact that R is finitely-generated that $\dim_k R_{\boldsymbol{\alpha}} < \infty$ for each $\boldsymbol{\alpha} \in \mathbf{N}^m$, and we define the *Hilbert function* $H(R, \boldsymbol{\alpha}) = \dim_k R_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathbf{N}^m$. If $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ let $\boldsymbol{\lambda}^{\boldsymbol{\alpha}} = \lambda_1^{\alpha_1} \cdots \lambda_m^{\alpha_m}$, and define the *Poincaré series* $F(R, \boldsymbol{\lambda}) = \sum_{\boldsymbol{\alpha}} H(R, \boldsymbol{\alpha})\boldsymbol{\lambda}^{\boldsymbol{\alpha}}$. It is well known that the formal power series $F(R, \boldsymbol{\lambda})$ represents a rational function of $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$. Let d be the *Krull dimension* $\dim R$ of R, i.e., the maximum number of elements of R which are algebraically independent over k. (Do not confuse the Krull dimension "dim" with the vector space dimension "$\dim_k$".) Suppose $\theta_1, \ldots, \theta_d$ are homogeneous elements of R of nonzero degree such that $\dim_k R/(\theta_1, \ldots, \theta_d) < \infty$, or equivalently, such that $\dim R/(\theta_1, \ldots, \theta_d) = 0$. Then $\theta_1, \ldots, \theta_d$ are called a *homogeneous system of parameters* (h.s.o.p.) for R. Again if we wish to emphasize that $\deg \theta_i \in \mathbf{N}^m$, $1 \leqslant i \leqslant d$, then we call $\theta_1, \ldots, \theta_d$ an $\mathbf{N}^m$-homogeneous system of parameters. If $m = 1$ then the Noether normalization lemma guarantees the existence of an h.s.o.p. Moreover, if k is infinite and R is generated by R_1, then we can choose each θ_i, $1 \leqslant i \leqslant d$, to have degree one. However, when $m > 1$ an h.s.o.p. usually will not exist; indeed, a crucial point of this paper concerns the existence of an h.s.o.p. in certain situations when $m > 1$ (Theorem 4.1). At any rate, suppose $\theta_1, \ldots, \theta_d$ is an h.s.o.p. for R. Let $S = R/(\theta_1, \ldots, \theta_d)$. Since $\theta_1, \ldots, \theta_d$ are homogeneous, S inherits from R the structure of an $\mathbf{N}^m$-graded k-algebra. We now say that R is *Cohen-Macaulay* if

$$F(R, \boldsymbol{\lambda}) = F(S, \boldsymbol{\lambda}) \prod_{i=1}^{d} (1 - \boldsymbol{\lambda}^{\deg \theta_i})^{-1}. \tag{1}$$

This is not the usual definition of a Cohen-Macaulay ring, but it is equivalent. For a reconciliation with the usual definition in terms of R-sequences, see [**24**]. It is important to realize that the question of whether or not R is Cohen-Macaulay is independent of the grading chosen for R, though this is not immediately obvious from (1). Thus once we know that R is Cohen-Macaulay, we know that (1) holds for whatever grading we choose for R. From the combinatorial point of view, the importance of Cohen-Macaulay rings R is that the much smaller ring S carries a lot of combinatorial information about R, in particular, the Hilbert function of R.

Now given a complex Δ on $V = \{x_1, \ldots, x_n\}$, associate with it a certain $\mathbf{N}$-graded k-algebra A_Δ as follows. Let $A = k[x_1, \ldots, x_n]$, the polynomial ring over k on the vertices of Δ. Let I_Δ be the ideal generated by all monomials $x_{i_1} x_{i_2} \cdots x_{i_s}$ with $i_1 < i_2 < \cdots < i_s$ and $\{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\} \notin \Delta$. Define a grading on $A_\Delta = A/I_\Delta$ by setting $\deg x_i = 1$. We say that Δ is a

Cohen-Macaulay complex (always with respect to the field k) if A_Δ is a Cohen-Macaulay ring. This is the algebraic definition of a Cohen-Macaulay complex.

To give the topological definition, recall that if $F \in \Delta$, then the *link* of F is the complex $\operatorname{lk} F = \{G \in \Delta: F \cap G = \varnothing, F \cup G \in \Delta\}$. In particular, $\operatorname{lk} \varnothing = \Delta$.

1.2 THEOREM. *Let Δ be a δ-complex and k a field. The following three conditions are equivalent.*

(i) *Δ is Cohen-Macaulay (over k).*

(ii) *For all $F \in \Delta$, $\tilde{H}_i(\operatorname{lk} F) = 0$ if $i < \dim(\operatorname{lk} F)$. (Here $\tilde{H}$ denotes reduced simplicial homology with coefficient field k.)*

(iii) *Let $X = |\Delta|$, the geometric realization of Δ, so that Δ is a triangulation of X. Then $\tilde{H}_i(X) = H_i(X, X - p) = 0$ for all $i < \delta$ and all $p \in X$. (Here $\tilde{H}$ denotes reduced singular homology and H relative singular homology, both over k.)*

The equivalence of (i) and (ii) above is a theorem of G. Reisner [**16**], while the equivalence of (ii) and (iii) is a purely topological result first explicitly proved by J. Munkres [**15**, Theorem 2.1]. A stronger result was later proved by Hochster [**10**, Theorem 4.1].

Let us remark that the following results are immediate consequences of Theorem 1.2: (a) every Cohen-Macaulay complex is pure, (b) a Cohen-Macaulay complex of dimension greater than zero is connected, and (c) a graph (= complex of dimension zero or one) is Cohen-Macaulay if and only if it has no edges or is connected.

Let $H(\Delta, m)$, $m \in \mathbf{N}$, denote the Hilbert function of A_Δ. It is easy to see [**22**, Proposition 3.2] that

$$H(\Delta, m) = \begin{cases} 1, & \text{if } m = 0, \\ \displaystyle\sum_{i=0}^{\delta} f_i \binom{m-1}{i}, & \text{if } m > 0, \end{cases} \tag{2}$$

where $(f_0, f_1, \ldots, f_\delta)$ is the f-vector of Δ. An immediate consequence of (2) (using [**1**, Theorem 11.4]) is the result $\dim A_\Delta = 1 + \dim \Delta = 1 + \delta$. Since $H(\Delta, m)$ is a polynomial in m for $m \geqslant 1$, it follows that there are integers $1 = h_0, h_1, \ldots, h_{\delta+1}$ such that

$$(1 - \lambda)^{1+\delta} F(A_\Delta, \lambda) = h_0 + h_1\lambda + \cdots + h_{1+\delta}\lambda^{1+\delta}.$$

The vector $\mathbf{h} = \mathbf{h}(\Delta) = (h_0, h_1, \ldots, h_{1+\delta})$ is called the *h-vector* of Δ. We wish to state a characterization analogous to Theorem 1.1 of the h-vector of a Cohen-Macaulay complex Δ. To do so, recall that a *multiset* M on a set S is a set with repeated elements belonging to S. More precisely, M is a function $S \to \mathbf{N}$, where $M(x)$ is regarded as the number of repetitions of $x \in S$. The *cardinality* of M is $\operatorname{card} M = \sum_{x \in S} M(x)$. A multiset $M'\colon S \to \mathbf{N}$ is a *submultiset* of M (denoted $M' \subset M$) if $M'(x) \leqslant M(x)$ for all $x \in S$. A *multicomplex* is a collection of multisets such that if $M \in \Lambda$ and $M' \subset M$, then $M' \in \Lambda$. The dimension and f-vector of a multicomplex are defined in the obvious way in analogy with complexes, i.e., $f_i = \operatorname{card}\{M \in \Lambda\colon \operatorname{card} M = i + 1\}$ and $\dim \Lambda = \max\{i\colon f_i \neq 0\}$. Any vector $(f_0, f_1, \ldots, f_\delta)$ which is the f-vector of a multicomplex is called an *M-vector*. We allow $f_{j+1} = f_{j+2} = \cdots = f_\delta = 0$ in an M-vector; if also $f_j \neq 0$ this means that the corresponding multicomplex Λ has dimension j.

In analogy with Theorem 1.1 we have the following result essentially due to Macaulay **[13]** (explaining our terminology "M-vector"). A common generalization of Theorems 1.1 and 1.3 appears in **[3]**, and an exposition of these results appears in **[9]**.

1.3 THEOREM. *Given positive integers f and i, write*

$$f = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geqslant j \geqslant 1$ (*exactly as in Theorem* 1.1); *and define*

$$f^{\langle i \rangle} = \binom{n_i + 1}{i + 1} + \cdots + \binom{n_j + 1}{j + 1},$$

with $0^{\langle i \rangle} = 0$. *Then the vector* $\mathbf{f} = (f_0, f_1, \ldots, f_\delta)$ *of nonnegative integers is an M-vector if and only if* $f_{i+1} \leqslant f_i^{\langle i+1 \rangle}$ *for* $0 \leqslant i \leqslant \delta - 1$. $\square$

We can now give the characterization **[23**, Theorem 6] of the h-vector of a Cohen-Macaulay complex.

1.4 THEOREM. *A vector* $(h_0, h_1, \ldots, h_{\delta+1})$ *is the h-vector of a Cohen-Macaulay complex of dimension* δ *if and only if* $h_0 = 1$ *and* $(h_1, h_2, \ldots, h_{\delta+1})$ *is an M-vector.* $\square$

2. Balanced complexes. We wish to introduce a class of Cohen-Macaulay complexes for which Theorem 1.4 can be considerably strengthened and refined. Recall that an *ordered partition* of a finite set V is a sequence $(V_1, \ldots, V_m)$ of nonvoid, pairwise disjoint subsets of V satisfying $V_1 \cup \cdots \cup V_m = V$.

DEFINITION. A *balanced complex of type* $(a_1, \ldots, a_m)$ is a pair (Δ, π) satisfying:

(i) Δ is a pure δ-complex on a vertex set V,

(ii) $\pi = (V_1, \ldots, V_m)$ is an ordered partition of V, and

(iii) for every maximal face $F \in \Delta$ and every $i \in [m]$, we have $\operatorname{card}(F \cap V_i) = a_i$. (Hence $a_1 + \cdots + a_m = \delta + 1$.)

A balanced complex of type $(1, 1, \ldots, 1)$ is called *completely balanced.* Note that a balanced complex of type $(\delta + 1)$ (i.e., with $m = 1$) is really nothing more than a pure δ-complex, since condition (iii) holds automatically. We could have altered our definition somewhat so that Δ need not be pure, but nothing significant is gained by doing so. In particular, we are primarily concerned with Cohen-Macaulay complexes, and these are always pure.

We now give some examples of completely balanced complexes. Let P be a poset (= partially ordered set) on a finite set V, and define $\Delta(P)$ to be the complex on V whose faces are the chains (= linearly ordered subsets) of P. We will use such terminology as "P is pure" or "P is Cohen-Macaulay" to mean the corresponding statement for $\Delta(P)$. Thus P is pure if and only if all maximal chains of P have the same length, and dim P is the length of the longest chain in P. Moreover, a *Cohen-Macaulay poset* P is one for which $\Delta(P)$ is a Cohen-Macaulay complex. Suppose now that P is pure of dimension δ. If $x \in V$, let $\rho(x)$ be the largest integer r for which there is a chain $x_1 < x_2 < \cdots < x_r = x$ in P. We call $\rho(x)$ the *rank* of x and ρ the *rank function* of P. (Some authors would call $\rho(x) - 1$ the rank or height of x.) If we set $V_i = \{x \in P\colon \rho(x) = i\}$, $1 \leqslant i \leqslant \delta + 1$, then clearly $\pi = (V_1, V_2, \ldots, V_{\delta+1})$ is an ordered partition of V and $(\Delta(P), \pi)$ is completely balanced. We call π the *standard* ordered partition of P. If Δ is any complex, let $Q = Q(\Delta)$ be the poset of nonvoid faces of Δ, ordered by inclusion. Then $\Delta(Q)$ is just the first barycentric subdivision of Δ. Hence any space X which possesses a finite pure triangulation possesses a completely balanced triangulation. There does not seem to be a nice characterization of completely balanced complexes, though a sufficient condition for Δ to be completely balanced is mentioned in **[5]**. On the other hand, one can characterize complexes Δ of the form $\Delta(P)$ for a finite poset P. Namely, it is necessary and sufficient that Δ satisfy the following:

(i) any minimal set of vertices which do not form a face of Δ has two elements (i.e., the ideal I_Δ is generated by quadratic monomials), and

(ii) let Γ be the 1-skelton of Δ. Then Γ must satisfy the well-known conditions of Gilmore and Hoffman **[8]**, Ghouilà-Houri **[7]**, or Gallai **[6]** for being a comparability graph.

3. Numerical invariants of balanced complexes. If (Δ, π) is a balanced complex of type $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ and if $\mathbf{b} = (b_1, b_2, \ldots, b_m) \in \mathbf{N}^m$, then define $f_\mathbf{b} = f_\mathbf{b}(\Delta, \pi)$ to be the number of faces $F \in \Delta$ for which $\operatorname{card}(F \cap V_i) = b_i$, $1 \leqslant i \leqslant m$. Note that $f_\mathbf{b} = 0$ unless $b_i \leqslant a_i$ for all i (written $\mathbf{b} \leqslant \mathbf{a}$). Note also that $f_i(\Delta) = \sum f_\mathbf{b}(\Delta, \pi)$, where the sum is over all vectors $\mathbf{b} \leqslant \mathbf{a}$ such that $\sum b_j = i + 1$. Hence the numbers $f_\mathbf{b}$ are a refinement of the numbers f_i.

If $S \subset [m]$, let (Δ_S, π_S) be the balanced complex defined as follows:
(i) $\Delta_S = \{F \in \Delta \colon F \cap V_i = \varnothing \text{ if } i \notin S\}$,
(ii) if

$$S = \{c_1, c_2, \ldots, c_r\} \text{ with } c_1 < c_2 < \cdots < c_r, \tag{3}$$

then $\pi_S = (V_{c_1}, V_{c_2}, \ldots, V_{c_r})$.

Hence (Δ_S, π_S) is balanced of type $(a_{c_1}, a_{c_2}, \ldots, a_{c_r})$.

For instance, if P is a pure poset of dimension δ and if π is the standard ordered partition of P, then $\Delta(P)_S = \Delta(P_S)$, where P_S is the poset obtained from P by removing all elements whose ranks do not belong to S. In particular, $P_{[\gamma]}$ is the so-called "rank γ upper-truncation of P". Note that in general if (Δ, π) has type $(a_1, \ldots, a_m)$, then $(\Delta_{[m]}, \pi_{[m]}) = (\Delta, \pi)$, and $\Delta_\varnothing = \varnothing$.

The following result is an immediate consequence of the definition of (Δ_S, π_S).

3.1 Proposition. *Let (Δ, π) be a balanced complex of type $(a_1, a_2, \ldots, a_m) = \mathbf{a}$. Let $S \subset [m]$, say $S = \{c_1, c_2, \ldots, c_r\}$ with $c_1 < c_2 < \cdots < c_r$, so that (Δ_S, π_S) is balanced of type $(a_{c_1}, a_{c_2}, \ldots, a_{c_r}) = \mathbf{a}'$. If $(b_{c_1}, \ldots, b_{c_r}) = \mathbf{b}' \leqslant \mathbf{a}'$, define $(b_1, \ldots, b_m) = \mathbf{b}$ by letting $b_i = 0$ if i is not one of the c_j. Then $f_{\mathbf{b}'}(\Delta_S, \pi_S) = f_{\mathbf{b}}(\Delta, \pi)$.* □

The significance of Proposition 3.1 is the following. If we know the numbers $f_{\mathbf{b}}(\Delta, \pi)$ for all $\mathbf{b}$, then we also know the numbers $f_{\mathbf{b}'}(\Delta_S, \pi_S)$ for all $S \subset [m]$ and all $\mathbf{b}'$.

We can refine the h-vector of a balanced δ-complex (Δ, π), where $\pi = (V_1, \ldots, V_m)$, just as we did the f-vector, as follows. We make the ring A_Δ into an $\mathbf{N}^m$-graded k-algebra by defining, for a vertex x of Δ, $\deg x$ to be the ith unit coordinate vector $\mathbf{e}_i \in \mathbf{N}^m$. Equivalently, $\boldsymbol{\lambda}^{\deg x} = \lambda_i$ if $x \in V_i$.

3.2 Proposition. *Let (Δ, π) be a balanced complex of type $\mathbf{a} = (a_1, \ldots, a_m)$, where $\pi = (V_1, \ldots, V_m)$, and let $H(A_\Delta, \mathbf{b})$ denote the Hilbert function of A_Δ with the above $\mathbf{N}^m$-grading. Then for all $\mathbf{b} = (b_1, \ldots, b_m) \in \mathbf{N}^m$,*

$$H(A_\Delta, \mathbf{b}) = \sum_{\mathbf{c}} f_{\mathbf{c}}(\Delta, \pi) \prod_{b_i > 0} \binom{b_i - 1}{c_i - 1},$$

where the sum is over all $(c_1, \ldots, c_m) = \mathbf{c} \leqslant \mathbf{a}$ such that $c_i = 0 \Leftrightarrow b_i = 0$.

Note that Proposition 3.2 reduces to (2) when $m = 1$.

Proof of Proposition 3.2. If $M = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a nonzero monomial appearing in A_Δ, then define the *support* of M by $\operatorname{supp} M = \{x_i \in V \colon \alpha_i > 0\}$. Note $\operatorname{supp} M \in \Delta$. Given $F \in \Delta$ with $c_i = \operatorname{card} F \cap V_i$, the number of monomials M satisfying $\operatorname{supp} M = F$ and $\deg M = \mathbf{b}$ is $\prod_{i\colon b_i > 0} \binom{b_i - 1}{c_i - 1}$, since there are $\binom{b_i - 1}{c_i - 1}$ monomials of degree b_i in c_i variables with each variable having positive exponent. Summing over all $F \in \Delta$ completes the proof. □

It is an immediate consequence of Proposition 3.2, or can be easily seen directly, that

$$F(A_\Delta, \boldsymbol{\lambda}) = \sum_{F \in \Delta} \prod_{x \in F} \lambda_{\rho(x)}(1 - \lambda_{\rho(x)})^{-1}, \tag{4}$$

where for each $x \in V$, $\rho(x)$ is defined by $x \in V_{\rho(x)}$.

Hence we obtain the following result, which will be of use later.

3.3 PROPOSITION. *Let* (Δ, π) *be a balanced complex of type* $(a_1, \ldots, a_m)$. *Then* $F(A_\Delta, \boldsymbol{\lambda})\prod_{i=1}^m (1 - \lambda_i)^{a_i}$ (*or equivalently* $F(A_\Delta, \boldsymbol{\lambda})\prod_{x\in F}(1 - \lambda_{\rho(x)})$ *for any maximal face* F *of* Δ) *is a polynomial* $P(A_\Delta, \boldsymbol{\lambda})$ *in* $\lambda_1, \ldots, \lambda_m$. *Moreover, the degree of* $P(A_\Delta, \boldsymbol{\lambda})$ *with respect to* λ_i *is no more than* a_i. *In particular, if* (Δ, π) *is completely balanced then every monomial appearing in* $P(A_\Delta, \boldsymbol{\lambda})$ *is squarefree.* $\square$

Now if $\mathbf{b} \in \mathbf{N}^m$ define $h_\mathbf{b} = h_\mathbf{b}(\Delta, \pi)$ to be the coefficient of $\boldsymbol{\lambda}^\mathbf{b}$ in the polynomial $F(A_\Delta, \boldsymbol{\lambda})\prod_{i=1}^m (1 - \lambda_i)^{a_i}$. Proposition 3.3 asserts that

$$h_\mathbf{b} = 0 \quad \text{unless } \mathbf{b} \leqslant \mathbf{a}. \tag{5}$$

Clearly $h_i(\Delta) = \sum h_\mathbf{b}(\Delta, \pi)$, where $\mathbf{b}$ ranges over all $(b_1, \ldots, b_m) \in \mathbf{N}^m$ satisfying $b_1 + \cdots + b_m = i$. Hence the numbers $h_\mathbf{b}$ are a refinement of the numbers h_i. Note that

$$\begin{aligned} F(A_\Delta, \boldsymbol{\lambda}) \prod_{i=1}^n (1 - \lambda_i)^{a_i} &= \left[\sum_{F \in \Delta} \prod_{x \in F} \lambda_{\rho(x)}(1 - \lambda_{\rho(x)})^{-1} \right] \prod_{i=1}^n (1 - \lambda_i)^{a_i} \\ &= \sum_{\mathbf{c}} \sum_{\substack{F \in \Delta \\ \operatorname{card}(F \cap V_i) = c_i}} \boldsymbol{\lambda}^\mathbf{c} \prod_{i=1}^n (1 - \lambda_i)^{a_i - c_i} \\ &= \sum_{\mathbf{c}} f_\mathbf{c}(\Delta, \pi)\boldsymbol{\lambda}^\mathbf{c} \prod_{i=1}^n (1 - \lambda_i)^{a_i - c_i} \end{aligned}$$

from which we get

$$h_\mathbf{b}(\Delta, \pi) = \sum_{\mathbf{c} \leqslant \mathbf{b}} f_\mathbf{c}(\Delta, \pi) \prod_{i=1}^m (-1)^{b_i - c_i} \binom{a_i - c_i}{b_i - c_i}. \tag{6}$$

3.4 EXAMPLE. (a) Let $\Delta = \langle abc, abe, acd, ade, bcg, bef, bfg, cdg, def, dfg \rangle$, so $|\Delta| \approx \mathbf{S}^2$. Let $V_1 = \{a, c, e, f, g\}$, $V_2 = \{b, d\}$ and $\pi = (V_1, V_2)$. Then (Δ, π) is balanced of type (2, 1). We have (writing f_{rs} for $f_{(r,s)}$) $f_{00} = 1, f_{10} = 5$, $f_{01} = 2, f_{20} = 5, f_{11} = 10, f_{21} = 10$. Also (writing $\boldsymbol{\lambda} = (\mu, \lambda)$),

$$\begin{aligned} F(A_\Delta, \boldsymbol{\lambda}) &= (1 - \mu)^2(1 - \lambda) + 5\mu(1 - \mu)(1 - \lambda) + 2\lambda(1 - \mu)^2 \\ &\quad + 5\mu^2(1 - \lambda) + 10\mu\lambda(1 - \mu) + 10\mu^2\lambda \\ &= 1 + 3\mu + \lambda + \mu^2 + 3\mu\lambda + \mu^2\lambda. \end{aligned}$$

Hence $h_{00} = h_{01} = h_{20} = h_{21} = 1, h_{10} = h_{11} = 3$.

(b) Suppose $\Delta = \langle ab, cd \rangle$ with $V_1 = \{a, c\}$, $V_2 = \{b, d\}$, and $\pi = (V_1, V_2)$. Then (Δ, π) is of type $(1, 1)$, i.e., is completely balanced. We have $f_{00} = 1, f_{10} = f_{01} = f_{11} = 2$. Also

$$\begin{aligned} F(A_\Delta, \boldsymbol{\lambda}) &= (1-\mu)(1-\lambda) + 2\lambda(1-\mu) + 2\mu(1-\lambda) + 2\mu\lambda \\ &= 1 + \mu + \lambda - \mu\lambda. \end{aligned}$$

Hence $h_{00} = h_{10} = h_{01} = 1, h_{11} = -1$.

In general it is difficult to obtain any intuition for the numbers $h_{\mathbf{b}}(\Delta, \pi)$. There are two special circumstances, however, in which they have additional interpretations. First define the *reduced Euler characteristic* $\tilde{\chi}(\Delta)$ of a complex Δ by $\tilde{\chi}(\Delta) = \chi(\Delta) - 1$, where $\chi(\Delta)$ is the usual Euler characteristic. Equivalently, $\tilde{\chi}(\Delta) = -1 + f_0(\Delta) - f_1(\Delta) + \dots$. In particular, $\tilde{\chi}(\varnothing) = -1$. Note that if the reduced homology of Δ satisfies $\tilde{H}_i(\Delta) = 0$ for $i < \delta = \dim \Delta$, then $(-1)^\delta \tilde{\chi}(\Delta) = \dim_k \tilde{H}_\delta(\Delta) \geqslant 0$.

3.5 PROPOSITION. *Let (Δ, π) be a balanced complex of type $(a_1, \dots, a_m)$. Let $S \subset [m]$, and define $\mathbf{b} = (b_1, \dots, b_m)$ by*

$$b_i = \begin{cases} a_i, & \text{if } i \in S, \\ 0, & \text{if } i \notin S. \end{cases}$$

Let (Δ_S, π_S) be the balanced complex defined by (3). *Then $h_{\mathbf{b}}(\Delta, \pi) = (-1)^\delta \tilde{\chi}(\Delta_S)$, where $\delta = b_1 + \cdots + b_m - 1 = \dim \Delta_S$.*

PROOF. Let $\mathbf{c} = (c_1, \dots, c_m) \leqslant \mathbf{b}$. For all i we have $\binom{a_i - c_i}{b_i - c_i} = 1$, since $b_i = a_i$ or $b_i = c_i = 0$. Hence from (6),

$$\begin{aligned} h_{\mathbf{b}}(\Delta, \pi) &= \sum_{\mathbf{c} \leqslant \mathbf{b}} f_{\mathbf{c}}(\Delta, \pi) \prod_{i=1}^{m} (-1)^{b_i - c_i} \\ &= (-1)^\delta \sum_{i=-1}^{\delta} f_i(\Delta_S)(-1)^i \quad (\text{with } f_{-1}(\Delta) = 1) \\ &= (-1)^\delta \tilde{\chi}(\Delta_S). \quad \square \end{aligned}$$

Note that if (Δ, π) is completely balanced then any $\mathbf{b} \leqslant \mathbf{a}$ satisfies the hypothesis of Proposition 3.5. Hence in the completely balanced case every $h_{\mathbf{b}}(\Delta, \pi)$ may be interepreted as a reduced Euler characteristic (up to sign).

For our second alternative interpretation of the numbers $h_{\mathbf{b}}(\Delta, \pi)$, we need to define the notion of "shellability." Our definition is slightly more general than that sometimes given, e.g., [**4**]. If Δ is a pure δ-complex, then a *shelling* of Δ is an ordering $F_1, F_2, \dots, F_r$ of the δ-faces of Δ (so $r = f_\delta(\Delta)$) such that for $1 \leqslant i \leqslant r - 1$, $(F_1 \cup F_2 \cup \cdots \cup F_i) \cap F_{i+1}$ is a nonvoid union of $(\delta - 1)$-faces of F_{i+1}. In exactly the same manner as McMullen's interpretation [**14**, p.

182] of $h_i(\Delta)$ when Δ is shellable (McMullen uses $g_{i-1}^{(d)}$ for our h_i), we obtain the following result.

3.6 PROPOSITION. *Let* (Δ, π) *be a balanced complex where* $\pi = (V_1, \ldots, V_m)$, *and let* $F_1, F_2, \ldots, F_r$ *be a shelling of* Δ. *For each* $i \in [r]$, *define* G_i *to be the unique minimal face of* F_i *which is not contained in* $F_1 \cup F_2 \cup \cdots \cup F_{i-1}$. *(In particular,* $G_1 = \varnothing$.*) Define* $\mathbf{b}(i) = (b_1(i), \ldots, b_m(i))$ *by* $b_j(i) = \operatorname{card} G_i \cap V_j$. *Then* $h_{\mathbf{b}}(\Delta, \pi)$ *is equal to the number of integers* $i \in [m]$ *for which* $\mathbf{b} = \mathbf{b}(i)$.

PROOF. Let $F_i(\boldsymbol{\lambda})$ be the Poincaré series for the pure subcomplex of Δ whose maximal faces are $F_1, F_2, \ldots, F_i$. Then by the definition of G_i and $\mathbf{b}(i)$,

$$F_{i+1}(\boldsymbol{\lambda}) = F_i(\boldsymbol{\lambda}) + \frac{\lambda^{\mathbf{b}(i)}}{\prod_{i=1}^m (1-\lambda_i)^{a_i}},$$

so that

$$F(A_\Delta, \boldsymbol{\lambda}) \prod_{i=1}^m (1-\lambda_i)^{a_i} = \sum_{i=1}^m \boldsymbol{\lambda}^{b(i)}.$$

The proof now follows from the definition of $h_{\mathbf{b}}(\Delta, \pi)$. $\square$

3.7 EXAMPLE. Let (Δ, π) be the balanced complex of Example 3.4(a). Then (writing *abc* for $\{a, b, c\}$, etc.) *abc*, *acd*, *ade*, *abe*, *cdg*, *dfg*, *def*, *bef*, *bfg*, *bcg* is a shelling of Δ. We have $G_1 = \varnothing$, $G_2 = d$, $G_3 = e$, $G_4 = be$, $G_5 = g$, $G_6 = f$, $G_7 = ef$, $G_8 = bf$, $G_9 = bg$, $G_{10} = bcg$. Since $V_1 = \{a, c, e, f, g\}$ and $V_2 = \{b, d\}$, we have $\mathbf{b}(1) = (0, 0)$, $\mathbf{b}(2) = (0, 1)$, $\mathbf{b}(3) = (1, 0)$, $\mathbf{b}(4) = (1, 1)$, $\mathbf{b}(5) = (1, 0)$, $\mathbf{b}(6) = (1, 0)$, $\mathbf{b}(7) = (2, 0)$, $\mathbf{b}(8) = (1, 1)$, $\mathbf{b}(9) = (1, 1)$, $\mathbf{b}(10) = (2, 1)$, and we obtain the same values for $h_{\mathbf{b}}(\Delta, \pi)$ as before.

Proposition 3.6 shows that $h_{\mathbf{b}}(\Delta, \pi) \geqslant 0$ when Δ is shellable. However, it is easy to see that all shellable complexes are Cohen-Macaulay, so that the inequality $h_{\mathbf{b}}(\Delta, \pi) \geqslant 0$ is subsumed and generalized by Theorem 4.4 below. For a survey of some aspects of the subject of shellability, see [**4**]. Examples of shellable complexes include: (i) the boundary complex of a simplicial convex polytope (but not necessarily a triangulation of a sphere), (ii) connected graphs and triangulations of 2-cells, (iii) the independent set complex and broken circuit complex [**23**, §7] of a finite matroid, and (iv) the complex $\Delta(P)$ where P is an admissible lattice in the sense of [**21**]. Example (iii) is due to Scott Provan and (iv) to Anders Bjørner.

There is a somewhat weaker condition than shellability which implies that $h_{\mathbf{b}}(\Delta, \pi) \geqslant 0$. If Δ is a complex and if $G \subset F$ are faces of Δ, then define the *interval* $[G,F]$ by $[G, F] = \{F' | G \subset F' \subset F\}$. An *upper partition* of a pure δ-complex Δ is a collection $[G_1, F_1], \ldots, [G_r, F_r]$ of intervals of Δ satisfying:

(i) $[G_i, F_i] \cap [G_j, F_j] = \varnothing$ if $i \neq j$,

(ii) $\Delta = [G_1, F_1] \cup \cdots \cup [G_r, F_r]$,

(iii) $\dim F_i = \delta$ for all $i \in [r]$ (so $r = f_\delta(\Delta)$).

A complex Δ possessing an upper partiton is said to be *partitionable*. It is easily seen that every shellable complex is partitionable. The converse is false. In fact, if $\Delta = \langle ab, cd, ce, de \rangle$ then Δ is partitionable but not even Cohen-Macaulay. An upper partition of Δ is given by $[\varnothing, ab]$, $[c, cd]$, $[d, de]$, $[e, ce]$. We do not know if every Cohen-Macaulay complex is partitionable. Now assume (Δ, π) is balanced with $\pi = (V_1, \ldots, V_m)$, and suppose $[G_1, F_1], \ldots, [G_r, F_r]$ is an upper partition of Δ. Then as a slight generalization of Proposition 3.6, it is easily shown that if $\mathbf{b} = (b_1, \ldots, b_m)$, then $h_\mathbf{b}(\Delta, \pi)$ is equal to the number of $j \in [r]$ for which card $G_j \cap V_i = b_i$ for all $i \in [m]$. Partitionable complexes were independently considered by Provan [25, Appendix 4].

4. Balanced Cohen-Macaulay complexes. A balanced complex (Δ, π) for which Δ is a Cohen-Macaulay complex is called a *balanced Cohen-Macaulay complex*. Our main aim is to give restrictions on the numbers $h_\mathbf{b}$ associated with a balanced Cohen-Macaulay complex which strengthens and refines Theorem 1.4. Our results are based on the following fundamental algebraic property of balanced complexes.

4.1 THEOREM. *Let (Δ, π) be a balanced complex of type $(a_1, \ldots, a_m)$, where $\pi = (V_1, \ldots, V_m)$. Let Δ_i stand for the complex $\Delta_{\{i\}}$ of* (3), *so that Δ_i is just the restriction of Δ to V_i. Set $A_i = A_{\Delta_i}$, and give this ring an* $\mathbf{N}$*-grading by defining* $\deg x = 1$ *for $x \in V_i$. Now give A_Δ the $\mathbf{N}^m$-grading* $\deg x = \mathbf{e}_i$ *if $x \in V_i$ (as defined preceding Proposition* 3.2). *Suppose Ψ_i is an $\mathbf{N}$-homogeneous system of parameters for A_i. Then $\Psi = \Psi_1 \cup \cdots \cup \Psi_m$ is an $\mathbf{N}^m$-homogeneous system of parameters for A_Δ.*

PROOF. If $\theta \in \Psi_i$ has degree $\alpha \in \mathbf{N}$ in A_i, then θ is $\mathbf{N}^m$-homogeneous in A_Δ, with $\deg \theta = \alpha \mathbf{e}_i \in \mathbf{N}^m$. Since $a_1 + a_2 + \cdots + a_m = \dim A_\Delta$, it remains only to show $\dim_k A_\Delta/(\Psi) < \infty$, where (Ψ) denotes the ideal generated by all $\theta \in \Psi$. Now A_Δ is a quotient ring of $A_1 \otimes_k A_2 \otimes_k \cdots \otimes_k A_m$, so

$$\dim_k A_\Delta / (\Psi) \leqslant \prod_{i=1}^{m} \dim_k A_i / (\Psi_i) < \infty.$$

This completes the proof. $\square$

4.2 COROLLARY. *Let (Δ, π) be completely balanced with $\pi = (V_1, \ldots, V_m)$. Let $\theta_i = \sum_{x \in V_i} x$. Then $\theta_1, \ldots, \theta_m$ is an $\mathbf{N}^m$-homogeneous system of parameters for A_Δ. Indeed,* $\deg \theta_i = \mathbf{e}_i$.

PROOF. We have $A_i = k[V_i]/I_i$, where I_i is generated by all products xx' such that $x, x' \in V_i$ and $x \neq x'$. From this it follows that the single element

θ_i is a system of parameters for A_i, and the proof follows from Theorem 4.1. $\square$

REMARK. When (Δ, π) is completely balanced, Corollary 4.2 gives an h.s.o.p. for A_Δ consisting of linear forms (i.e., of **N**-degree one). There is a more general result which gives a necessary and sufficient condition for a set of linear forms to be an **N**-h.s.o.p. for *any* A_Δ. Let $d = \dim A_\Delta = 1 + \dim \Delta$, and let

$$\theta_i = \sum_{j=1}^{n} \alpha_{ij} x_j, \qquad \alpha_{ij} \in k,\ 1 \leqslant i \leqslant d,$$

be a set of d linear forms in A_Δ. Then $\theta_1, \ldots, \theta_d$ is an **N**-h.s.o.p. if and only if for every $F \in \Delta$ (equivalently, for every maximal $F \in \Delta$) the $d \times (\text{card } F)$ matrix (α_{ij}), where $1 \leqslant i \leqslant d$ and $x_j \in F$, has rank equal to card F. We omit the proof.

Theorem 4.1 allows an easy proof of the next result.

4.3 THEOREM. *Let (Δ, π) be a balanced Cohen-Macaulay complex of type $(a_1, \ldots, a_m)$, and let $S \subset [m]$. Then (Δ_S, π_S) is a balanced Cohen-Macaulay complex of type $(a_{c_1}, a_{c_2}, \ldots, a_{c_r})$, where $S = \{c_1, c_2, \ldots, c_r\}$ and $c_1 < c_2 < \cdots < c_r$.*

PROOF. We only need to prove that Δ_S is Cohen-Macaulay. By Theorem 1.2, the desired result is of a purely topological nature. An almost equivalent result was first proved by J. Munkres [**15**, Theorem 6.4] using topological methods, and his proof straightforwardly extends to Theorem 4.3. However, it may be of interest to give a simple alternative proof based directly on the definition (1) of a Cohen-Macaulay ring.

Let A_Δ have the usual $\mathbf{N}^m$-grading defined by $\deg x = \mathbf{e}_i$ if $x \in V_i$. Let $\Psi = \psi_1 \cup \cdots \cup \Psi_m$ be a homogeneous system of parameters of the type described by Theorem 4.1. Then by (1),

$$F(A_\Delta, \boldsymbol{\lambda}) \prod_{\theta \in \Psi} (1 - \boldsymbol{\lambda}^{\deg \theta}) = F(A_\Delta / (\Psi), \boldsymbol{\lambda}). \tag{7}$$

Since Δ_S consists of those faces $F \in \Delta$ for which $x \in V_{c_1} \cup \cdots \cup V_{c_r}$ whenever $x \in F$, it follows that $F(A_{\Delta_S}, \boldsymbol{\lambda})$ is obtained from $F(A_\Delta, \boldsymbol{\lambda})$ by setting $\lambda_i = 0$ if $i \notin S$ and then substituting λ_i for λ_{c_i}. Let $\Psi_S = \bigcup_{i \in S} \Psi_i$. By Theorem 4.1, Ψ_S is an h.s.o.p. for A_{Δ_S}. If deg (resp. $\deg_S$) denotes degree in A_Δ (resp. A_{Δ_S}), it follows that $\prod_{\theta \in \Psi_S}(1 - \boldsymbol{\lambda}^{\deg_S \theta})$ is obtained from $\prod_{\theta \in \Psi}(1 - \boldsymbol{\lambda}^{\deg \theta})$ by the same substitution as above. Now note that $A_{\Delta_S}/(\Psi_S) = A_\Delta/(\Psi, X)$, where X consists of all $x \in V$ such that $x \notin V_i$ for any $i \in S$. Since $A_\Delta/(\Psi)$ is $\mathbf{N}^m$-graded, a k-basis for $A_\Delta/(\Psi, X)$ consists of those monomials in $A_\Delta/(\Psi)$ whose support lies in $V_{c_1} \cup \cdots \cup V_{c_r}$. The remaining monomials in $A_\Delta/(\Psi)$ are zero modulo X. Hence $F(A_\Delta/(\Psi, X), \boldsymbol{\lambda})$ is obtained from $F(A_\Delta/(\Psi), \lambda)$ by making once again the same substitution $\lambda_i \to 0$ if $i \notin S$ and then $\lambda_{c_i} \to \lambda_i$. Hence when we make this substitution in (7) we obtain

$$F(A_{\Delta_S}, \lambda) \prod_{\theta \in \Psi_S} (1 - \lambda^{\deg_S \theta}) = F(A_{\Delta_S}/(\Psi_S), \lambda).$$

By (1), it follows that A_{Δ_S} is Cohen-Macaulay. □

We are now in a position to discuss the h-vectors of balanced Cohen-Macaulay complexes.

4.4 THEOREM. *Let* (Δ, π) *be a balanced Cohen-Macaulay complex of type* $\mathbf{a} = (a_1, \ldots, a_m)$, *with* $\pi = (V_1, \ldots, V_m)$. *Let* $v_i = \operatorname{card} V_i$, *and let* T *be a set with* $\sum_{i=1}^m (v_i - a_i)$ *elements, say* $T = \{y_{ij}\colon 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant v_i - a_i\}$. *Then there exists a multicomplex* Λ *on* T *with the following property: For every* $(b_1, \ldots, b_m) \in \mathbf{N}^m$, *the number of* $M \in \Lambda$ *satisfying*

$$\sum_{j=1}^{v_i - a_i} M(y_{ij}) = b_i, \quad \textit{for all } i \in [m], \tag{8}$$

is equal to $h_{\mathbf{b}}(\Delta, \pi)$. *Hence by* (5),

$$\sum_{j=1}^{v_i - a_i} M(y_{ij}) \leqslant a_i, \quad \textit{for all } i \in [m]. \tag{9}$$

Before proving this result, we first discuss its significance. According to Theorem 1.4, the h-vector of a Cohen-Macaulay complex Δ is the f-vector of some multicomplex Λ. Theorem 4.4 asserts that Λ must have certain special properties when (Δ, π) is balanced of type $(a_1, \ldots, a_m)$. It follows from (9) that $M(y_{ij}) \leqslant a_i$ for all $i \in [m]$. Thus each $y_{ij} \in T$ has a restriction as to its multiplicity in any $M \in \Lambda$. In general, given a vector $\mathbf{c} = (c_1, c_2, \ldots, c_r)$ where each c_i is a positive integer or ∞, there is a characterization analogous to Theorems 1.1 and 1.3 for the f-vector of a multicomplex Λ on a set $S = \{y_1, \ldots, y_r\}$ such that $M(y_i) \leqslant c_i$ for all $i \in [r]$. This characterization is essentially due to Clements and Lindström [3], although an explicit numerical statement first appeared in [2] and is restated succinctly in [9]. Note that Theorem 1.1 corresponds to the case $\mathbf{c} = (1, 1, \ldots, 1)$ and Theorem 1.3 to the case $\mathbf{c} = (\infty, \infty, \ldots, \infty)$. At any rate, Theorem 4.4 shows that Λ must satisfy the characterization with $v_i - a_i$ of the c_j's equal to a_i. But Theorem 4.4 actually asserts a much stronger result, viz., the elements of certain subsets of T cannot have their *combined* multiplicities greater than a_i. Moreover, Theorem 4.4 places a restriction not merely on the ordinary h-vector of Δ, but on the "refined" numbers $h_{\mathbf{b}}(\Delta, \pi)$. Unfortunately the condition which Theorem 4.4 places on $h_{\mathbf{b}}(\Delta, \pi)$ is not strong enough to characterize the number $h_{\mathbf{b}}(\Delta, \pi)$ when (Δ, π) is balanced of some fixed type $(a_1, \ldots, a_m)$

unless $m = 1$ (Theorem 1.4), so for $m > 1$ we do not have as complete a result as Theorem 1.4.

The completely balanced case of Theorem 4.4 is of special interest and deserves a separate statement. When (Δ, π) is completely balanced, (9) implies that each $M(y_i) \leqslant 1$. In other words, Λ is actually a complex, not just a multicomplex, and we obtain:

4.5 COROLLARY. *Let (Δ, π) be a completely balanced Cohen-Macaulay complex. Then the h-vector of Δ is the f-vector of some complex Λ and therefore satisfies Theorem* 1.1. (*In other words,* $\mathbf{h}(\Delta)$ *is not merely an M-vector, but also a K-vector.*) *Even more strongly, there is an ordered partition $(T_1, \ldots, T_m)$ of the vertex set T of Λ such that* card $T_i \cap F \leqslant 1$ *for all $F \in \Lambda$, $i \in [m]$. Equivalently, the* 1*-skeleton of Λ can be m-colored in the usual graph-theoretical sense. Moreover, Λ can be chosen so that for any* 0-1 *vector* $\mathbf{b} = (b_1, \ldots, b_m)$, *there are exactly $h_{\mathbf{b}}(\Delta, \pi)$ faces $F \in \Lambda$ satisfying: $T_i \cap F = \varnothing$ if and only if $b_i = 0$.*
□

As an example to show that the conditions on Λ given by Corollary 4.5 are not sufficient to characterize the h-vector of a completely balanced Cohen-Macaulay complex, let Λ be the complex on $T = \{x_1, \ldots, x_7\}$ with maximal faces $\{x_1, x_2\}$, $\{x_3, x_4\}$, $\{x_5\}$, $\{x_6\}$, $\{x_7\}$. Let $T_1 = \{x_1, x_3, x_5, x_6, x_7\}$, $T_2 = \{x_2, x_4\}$. Then (Δ, π) would satisfy $h_{00} = 1$, $h_{10} = 5$, $h_{01} = 2$, $h_{11} = 2$. In particular, (Δ, π) would be of type $(1, 1)$ (i.e., a bipartite graph) with 6 vertices and 10 edges, and no such graph exists.

We remark in passing one additional property of completely balanced Cohen-Macaulay complexes (Δ, π). Namely, π is uniquely determined up to order. In other words, if (Δ, σ) is also completely balanced, then the entries V_i of π are a permutation of those of σ. Thus the 1-skeleton of Δ is a so-called "uniquely m-colorable graph." The proof is omitted. The corresponding statement for arbitrary completely balanced complexes is false, as shown by the example $\Delta = \langle ab, cd \rangle$.

PROOF OF THEOREM 4.4. If K is an extension field of k, then the ring $A_\Delta \otimes_k K$ has the same Hilbert function as A_Δ and is Cohen-Macaulay if and only if A_Δ is Cohen-Macaulay. Hence we may assume k is infinite. By Theorem 4.1 there is an $\mathbf{N}^m$-homogeneous system of parameters $\theta_1, \ldots, \theta_d$ $(d = a_1 + \cdots + a_m = 1 + \dim \Delta)$ for A_Δ, and our assumption that k is infinite implies we can choose them so that exactly a_i of them have degree $\mathbf{e}_i \in \mathbf{N}^m$. Let $B_\Delta = A_\Delta/(\theta_1, \ldots, \theta_d)$. Hence by (1) and our assumption that (Δ, π) is Cohen-Macaulay, we have

$$F(A_\Delta, \lambda) \prod_{i=1}^{m} (1 - \lambda_i)^{a_i} = F(B_\Delta, \lambda).$$

Thus by the definition of $h_{\mathbf{b}}(\Delta, \pi)$, we have

$$F(B_\Delta, \boldsymbol{\lambda}) = \sum_{\mathbf{b}} h_{\mathbf{b}}(\Delta, \pi)\boldsymbol{\lambda}^{\mathbf{b}}. \tag{10}$$

Now suppose $V_i = \{x_{i1}, \ldots, x_{i\nu_i}\}$. Let y_{ij} denote the image in B_Δ of x_{ij}. Since the a_i parameters of degree $\mathbf{e}_i$ are linearly independent, it follows that B_Δ is generated as a k-algebra by the elements y_{ij}, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant \nu_i - a_i$. Hence B_Δ has a k-basis consisting of monomials in these y_{ij}. A simple argument due to Macaulay [**13**] and also given in [**24**, Theorem 2.1] shows that we can pick this k-basis to be a multicomplex Λ on the set of y_{ij}. By (10), it follows that the number of $M \in \Lambda$ satisfying (8) is $h_{\mathbf{b}}(\Delta, \pi)$, completing the proof. $\square$

5. Posets and Möbius functions. In the special case that $\Delta = \Delta(P)$ for some poset P, our previous results are closely related to certain well-known concepts associated with P. In this section we will sketch this relation.

Let P be a (finite) pure poset with rank function ρ, i.e., $\rho(x)$ is the cardinality of a saturated chain of P with maximum element x. Let $\hat{P}$ denote the poset obtained by adjoining a minimum element $\hat{0}$ and maximum element $\hat{1}$ to P, i.e., $\hat{0} < x < \hat{1}$ for all $x \in P$. Let μ denote the Möbius function of $\hat{P}$, as defined in [**17**]. Thus, μ is a function from $\{(x, y) \in \hat{P} \times \hat{P} \colon x \leqslant y\}$ to $\mathbf{Z}$ satisfying

$$\mu(x, x) = 1 \quad \text{for all } x \in \hat{P},$$

$$\sum_{x \leqslant y \leqslant z} \mu(x, y) = 0 \quad \text{for all fixed pairs } x < z \text{ in } \hat{P}.$$

We also write $\mu(x)$ for $\mu(\hat{0}, x)$ and $\mu(P)$ for $\mu(\hat{0}, \hat{1})$.

If $\dim P = \delta$ (i.e., every maximal chain of P has cardinality $\delta + 1$) and $S \subset [\delta + 1]$, define $\alpha(P, S)$ to be the number of chains $x_1 < x_2 < \cdots < x_s$ in P such that $\{\rho(x_1), \ldots, \rho(x_s)\} = S$. Thus $\alpha(P, \varnothing) = 1$, $\alpha(P, \{i\})$ is the number of elements in P of rank i, and $\alpha(P, [\delta + 1])$ is the number of maximal chains of P. Equivalently, $\alpha(P, S)$ is the number of maximal chains of the poset

$$P_S = \{x \in P \colon \rho(x) \in S\},$$

or equivalently, the number of maximal chains of the poset $\hat{P}_S$ consisting of P_S with $\hat{0}$ and $\hat{1}$ adjoined. Now for $S \subset [\delta + 1]$ define

$$\beta(P, S) = \sum_{T \subset S} (-1)^{\operatorname{card}(S - T)} \alpha(P, T).$$

Equivalently, $\alpha(P, S) = \sum_{T \subset S} \beta(P, T)$. It is an immediate consequence of "Philip Hall's theorem" [**17**, Proposition 6] that

$$\beta(P, S) = (-1)^{1 + \operatorname{card} S} \mu(P_S). \tag{11}$$

The numbers $\alpha(P, S)$ and $\beta(P, S)$ were studied for various classes of posets in **[18]**, **[19]**, **[21]**, where they were shown to have many interesting properties. For instance, if $\hat{P}$ is a semimodular lattice then $\beta(P, S) \geqslant 0$ for all $S \subset [\delta + 1]$. It comes as no surprise that this result may be regarded as a consequence of the fact that semimodular lattices are Cohen-Macaulay, and Cohen-Macaulay posets seem like the right context for obtaining such results.

Many well-known numerical invariants of (pure) posets $\hat{P}$ can be expressed in terms of the basic numbers $\beta(P, S)$. For instance, the "Whitney number of the second kind" $W_i(\hat{P})$ is defined by $W_i(\hat{P}) = \operatorname{card}\{x \in P\colon \rho(x) = i\}$ and is clearly given by

$$W_i(\hat{P}) = \alpha(P, \{i\}) = \beta(P, \{i\}) + 1.$$

The "Whitney number of the first kind" $w_i(\hat{P})$ is defined by $w_i(\hat{P}) = \sum_{\rho(x)=i} \mu(x)$, and it is not hard to see that

$$(-1)^i w_i(\hat{P}) = \beta(P, [i-1]) + \beta(P, [i]).$$

Another commonly studied invariant is

$$(-1)^\delta \sum_{\rho(x)=i} \mu(x)\mu(x, \hat{1}) = \beta(P, [\delta+1] - \{i\}) + \beta(P, [\delta+1]).$$

A related invariant of $\hat{P}$ is the *zeta polynomial* **[20**, §3**]**. If $m \in \mathbf{N}$, define $Z(P, m)$ to be the number of chains $\hat{0} = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_m = \hat{1}$ in $\hat{P}$. Thus $Z(P, 0) = 0$, $Z(P, 1) = 1$, and $Z(P, 2) = \operatorname{card} \hat{P}$. It is easily seen that $Z(P, m)$ is a polynomial function of m of degree $\delta + 2$. It follows that there are constants $e_0, \ldots, e_{\delta+1}$ and $h_0, \ldots, h_{\delta+1}$ such that $Z(P, m) = \sum_{i=1}^{\delta+2} e_{i-1}\binom{m}{i}$ and

$$(1-\lambda)^{\delta+3} \sum_{m=0}^{\infty} Z(P, m)\lambda^m = \lambda\left(h_0 + h_1\lambda + \cdots + h_{\delta+1}\lambda^{\delta+1}\right). \quad (12)$$

It is not hard to see that

$$e_i = \sum_{\substack{S \subset [\delta+1] \\ \operatorname{card} S = i}} \alpha(P, S) \quad \text{and} \quad h_i = \sum_{\substack{S \subset [\delta+1] \\ \operatorname{card} S = i}} \beta(P, S).$$

We now discuss the relationship between the numbers $\beta(P, S)$ and the complex $\Delta(P)$. We have already noted in §2 that when P is pure of dimension δ, there is a standard ordered partition $\pi = (V_1, \ldots, V_{\delta+1})$ defined by $V_i = \{x \in P\colon \rho(x) = i\}$ which makes $(\Delta(P), \pi)$ completely balanced. Now Philip Hall's theorem is equivalent to the formula

$$\mu(x, y) = \tilde{\chi}(\Delta(x, y)), \qquad x < y,$$

where $(x, y) = \{z \in P\colon x < z < y\}$ (see **[17**, p. 346**]**). In particular, $\mu(P_S) = \tilde{\chi}(P_S)$. It is then an immediate consequence of Proposition 3.5 and (11) (or otherwise) that $h_{\mathbf{b}}(\Delta(P), \pi) = \beta(P, S)$, where $\mathbf{b} = (b_1, \ldots, b_{\delta+1})$, $b_i = 1$ if $i \in S$, $b_i = 0$ if $i \notin S$. Hence we see that if P is a Cohen-Macaulay poset, then the numbers $\beta(P, S) = h_{\mathbf{b}}(\Delta(P), \pi)$ satisfy the stringent requirements of Corollary 4.5. In particular, we obtain new restrictions on the Whitney numbers of the first and second kind of a Cohen-Macaulay poset.

Now note that if (x, y) is an open interval of P, then (x, y) is the link of a face of $\Delta(P)$ of the form $x_1 < x_2 < \cdots < x_r = x < y = y_s < y_{s-1} < \cdots < y_1$, where $\rho(x_i) = i$ and $\rho(y_j) = \delta + 2 - j$. Hence by Theorem 1.2, (x, y) is also a Cohen-Macaulay poset. The fact that $\beta(P, S) \geqslant 0$ for any Cohen-Macaulay poset P then implies the following: Let P be a Cohen-Macaulay poset, and let (x, y) be an open interval of length l in $\hat{P}$. Then $(-1)^l \mu(x, y) \geqslant 0$, where μ is the Möbius function of $\hat{P}$. Indeed, $(-1)^l \mu(x, y)$ is the lth Betti number (with respect to the field k) of the complex $\Delta(x, y)$. In the terminology of poset theory, the Möbius function of $\hat{P}$ "alternates in sign." Since by Theorem 4.3 each P_S is Cohen-Macaulay when P is, it follows that the Möbius function of each P_S also alternates in sign [**23**, §8].

It should also be pointed out that when P is any pure poset with the standard ordered partition π, then the numbers $\alpha(P, S)$ are identical to $f_{\mathbf{b}}(\Delta(P), \pi)$, with $b_i = 1$ if $i \in S$, $b_i = 0$ if $i \notin S$. The zeta polynomial $Z(P, m)$ is just the function $H(\Delta, m - 1)$ of (2), and the vector $(h_0, h_1, \ldots, h_{\delta+1})$ of (12) is just the h-vector of $\Delta(P)$.

There are two main classes of Cohen-Macaulay posets known: (i) semimodular lattices, or more generally, semimodular posets. (A poset P is *semimodular* if for every closed interval I of $\hat{P}$, and for every x, y in I such that x and y cover some element u of I, there is a $v \in I$ which covers both x and y. If $\hat{P}$ is a lattice, then it suffices to consider only the case $I = \hat{P}$.) (ii) The lattice of faces of a regular cell complex (e.g., a simplicial complex or the boundary complex (not necessarily simplicial) of a convex polytope) whose underlying space X satisfies Theorem 1.2(iii). In addition, if P and Q are Cohen-Macaulay, then so is their ordinal sum $P \oplus Q$. ($P \oplus Q$ is the partial order on the disjoint union of P and Q defined by $x \leqslant y$ in $P \oplus Q$ if (i) $x \leqslant y$ in P, or (ii) $x \leqslant y$ in Q, or (iii) $x \in P$ and $y \in Q$.) Indeed, $\Delta(P \oplus Q)$ is just the *join* $\Delta(P) * \Delta(Q)$, and $A_{\Delta(P \oplus Q)} = A_{\Delta(P)} \otimes_k A_{\Delta(Q)}$.

When $\hat{P}$ is an admissible lattice there is a combinatorial interpretation of the numbers $\beta(P, S)$ which implies they are nonnegative [**21**]. To give the reader the flavor of this result, we mention the interpretation of $\beta(P, S)$ when $\hat{P}$ is the lattice of subspaces of an n-dimensional vector space over $GF(q)$. In this case,

$$\beta(P, S) = \sum_{\pi} q^{i(\pi)}, \tag{13}$$

where the sum is over all permutations $\pi = (a_1, \ldots, a_n)$ of $[n]$ satisfying $S = \{i\colon a_i > a_{i+1}\}$, and where $i(\pi) = \text{card}\{(i,j)\colon i < j \text{ and } a_i > a_j\}$. This suggests that admissible lattices are Cohen-Macaulay, and indeed this has been shown by Anders Bjørner (to be published). It would be of considerable interest to obtain results analogous to (13) for other classes of Cohen-Macaulay posets, such as the semimodular posets which are not lattices, or the lattice of faces of a convex polytope.

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